

# HOMOTOPY PERTURBATIONS METHOD: THEORETICAL ASPECTS & APPLICATIONS

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## ABSTRACT

The application of the homotopy perturbation method (HPM) in two different research's area has been proposed in this paper. First, the HPM has been used for approximate solving of the well-known implicit equation for electrostatic surface potential of MOSFET transistor. The approximate analytical solution obtained in this case has relative simple mathematical form, and simultaneously high degree of accuracy. Next, HPM has been applied in determination of the invariant measures (IMs) of the non-linear dynamical systems with chaotic behavior. The convergence and efficiency of this method have been confirmed and illustrated in some characteristic examples of chaotic mappings.

**Keywords.** Homotopy perturbations, convergence, MOSFET modeling, invariant measures, chaotic maps.

## 1. INTRODUCTION

The homotopy perturbation method (HPM) belongs to the general and powerful techniques for solving the nonlinear equations of various kinds. As a combination of the well-known homotopy method in topology and classic perturbation techniques, the HPM was first introduced in pioneer works of He (1999, 2000, 2003, 2006, 2008). After that, the extensive development and application of this method to various fields of science researches has been started. For instance, the HPM has been successfully applied for obtaining analytic or approximate solutions of nonlinear differential and partial-differential equations (Biazar et al. 2009; El-Sayed et al. 2012; Gadallah & Elzaki 2017), as well as Fredholm and Volterra integral equations (Hetmaniok et al. 2012, 2013; Dong et al. 2013). Furthermore, this method also has found significant application in solving many kinds of real based problems, mainly in the physical sciences (Zeb et al. 2014; Kevkić et al. 2017, 2018).

Let us emphasize that the HPM is increasingly being used also in scientific fields, such as, for example, environmental protection and epidemiology (Khan et al. 2014; Adamu et al. 2017). Thus, the wide variety of applications indicates to the flexibility and importance of this method. Finally, let us point out that HPM has been improving, developing and modifying, until to the present time. Consequently, today exist various solver techniques that are based, to a greater or lesser extent, on the basic HPM assumptions (Noor & Khan 2012; Zhang et al. 2015; Tripathi & Mishra 2016; Bota & Caruntu 2017). Here is given a brief theoretical background about the HPM and some sufficient conditions to its convergence. Further, the HPM has been applied in two research's area, where still not observed it's any significant application.

As a novel approach, the HPM technique firstly has been applied for solving of the implicit relation between the electrostatic surface potential of an n-channel MOSFET transistor with terminal voltages. Approximate surface potential obtained in this way shows relative simple mathematical form, and at the same times a high degree of accuracy. Indeed, these properties of solution have crucial importance from the physical as well as design point of view. Further, the application of the HPM in determination of the invariant measures (IMs) of the non-linear dynamical

systems with chaotic behavior has been investigated. For this purpose, the convergence and efficiency of the HPM has been confirmed and illustrated with some characteristic examples of chaotic mappings.

## 2. METHODOLOGY OF THE HPM

For simple illustration of the basic concepts of HPM, we consider the following nonlinear equation:

$$\mathcal{N}[f(x)] = 0. \quad (1)$$

Here,  $f: \mathbb{R} \rightarrow \mathbb{R}$  is unknown function and  $\mathcal{N}(\cdot)$  is nonlinear operator defined on some functional domain  $\Omega$ . In solving Eq. (1) by using the HPM technique is assumed the introduction of *homotopy*  $\mathcal{H}: \Omega \times [0, 1] \rightarrow \mathbb{R}$  such that, for an arbitrary  $u \in \Omega$ :

$$\mathcal{H}[u, 0] = \mathcal{L}[u], \quad \mathcal{H}[u, 1] = \mathcal{N}[u], \quad (2)$$

where  $\mathcal{L}[u]$  is a linear operator, defined on the same domain  $\Omega$ . More precisely, if denote with  $p \in [0, 1]$  the so-called *embedding parameter*, then homotopy  $\mathcal{H}$  can be defined as a function

$$\mathcal{H}[u, p] = (1 - p) \mathcal{L}[u] + p \mathcal{N}[u], \quad (3)$$

which obviously satisfies the both of Eqs. (2). According to this, for the unknown function  $f(x)$ , we can form the so-called *homotopy equation*:

$$\mathcal{H}[f(x), p] = 0. \quad (4)$$

When  $p = 0$ , Eqs. (2)-(4) give the following linear equation:

$$\mathcal{L}[f(x)] = 0, \quad (5)$$

for which is assumed that has a unique solution  $f_0(x)$ , usually called *initial solution* or *initial approximation*. On the other hand, when  $p = 1$  the same Eqs. (2)-(4) obviously give the nonlinear Eq. (1).

The basic assumption of HPM is that general solution  $F(x, p)$  of the homotopy Eq. (4) can be expressed as the power series in  $p$ :

$$F(x, p) = \sum_{k=0}^{+\infty} p^k f_k(x). \quad (6)$$

From here, we get immediately  $F(x, 0) = f_0(x)$  as the initial solution of linear Eq. (5), i.e. as the solution of homotopy Eq. (4) when  $p = 0$ . On the other hand, the solution of the “main” Eq. (1), or equivalently Eq. (4) when  $p = 1$ , will be:

$$f(x) = \lim_{p \rightarrow 1^-} F(x, p) = \sum_{k=0}^{\infty} f_k(x). \quad (7)$$

on the condition of convergence the series in Eq. (7). In this case, according to Abel theorem, immediately follows:

**Theorem 1.** *If series  $\sum_{k=0}^{\infty} f_k(x)$  converges, then function  $F(x, p)$  is continuous from the left at  $p = 1$ , i.e. Eq. (7) holds.*

Notice that, if the conditions of the previous theorem are fulfilled, then series  $\sum_{k=0}^{\infty} f_k(x)$  represents the solution of the homotopy Eq. (4) when  $p = 1$ , i.e., it is solution of the nonlinear Eq. (1). Moreover, the solution  $f(x)$  of Eq. (1) can be estimated by the so-called *HPM-approximations*:

$$\begin{aligned} \widehat{f}_0(x) &= f_0(x), \\ \widehat{f}_k(x) &= \widehat{f}_{k-1}(x) + f_k(x) = \sum_{j=1}^k f_j(x), \quad k = 1, 2, \dots \end{aligned} \quad (8)$$

Obviously, series  $\sum_{k=0}^{\infty} f_k(x)$  is a solution of Eq. (1) if and only if the series  $\{\widehat{f}_k(x)\}$  converges to the unknown function  $f(x)$ . Sufficient conditions for this convergence can be also given by the following statement, which is a special case of Banach fixed point theorem:

**Theorem 2.** *Let  $\Omega$  is Banach space with sup-norm  $\|\cdot\|$  and the series  $\{f_k(x)\}$  defined on  $\Omega$ . If for some  $\alpha \in (0, 1)$  and  $k \geq 1$  the inequality  $\|f_k(x)\| \leq \alpha \|f_{k-1}\|$  holds, then series  $\{\widehat{f}_k(x)\}$ , defined by Eqs. (8), uniformly converges to the unique solution  $f(x)$  of Eq. (1).*

*Proof.* According to assumptions of the theorem and Eqs. (8), for an arbitrary  $k, m > 0$  we have that:

$$\begin{aligned} \|\widehat{f}_{k+m} - \widehat{f}_k\| &\leq \|\widehat{f}_{k+m} - \widehat{f}_{k+m-1}\| + \dots + \|\widehat{f}_{k+1} - \widehat{f}_k\| \\ &\leq \|f_{k+m}\| + \dots + \|f_{k+1}\| \\ &\leq (\alpha^m + \dots + \alpha) \|f_k\| \\ &\leq \alpha^{k+1} \frac{1 - \alpha^m}{1 - \alpha} \|f_0\|. \end{aligned}$$

Thus,  $\lim_{k, m \rightarrow \infty} \|\widehat{f}_{k+m} - \widehat{f}_k\| = 0$ , i.e.  $\{\widehat{f}_k(x)\}$  is a Cauchy sequence in

Banach space  $\Omega$ . It implies that  $\{\widehat{f}_k(x)\}$  is uniformly convergent, and its limit is uniquely determined by Eq. (7).  $\square$

In following will be describe some practical applications of the aforementioned HPM methodology.

### 3. MODELING SURFACE POTENTIAL IN MOSFET TRANSISTORS

The most of MOSFET transistor models are based on charge sheet approximation and the incrementally linear relationship between the inversion charge density and the surface potential (van Langevelde & Klaassen 2000; Chen & Gildenblat 2001). Their combination gives following implicit relation between the surface potential  $\psi_s$  and gate voltage  $V_G$ :

$$V_G - V_{FB} - \psi_s = \gamma \sqrt{\psi_s + u_T \exp\left(\frac{\psi_s - 2\phi_F - V_{ch}}{u_T}\right)}. \quad (9)$$

where  $V_{FB}$  is the flat band voltage,  $u_T$  is the thermal voltage,  $\phi_F$  is Fermi potential,  $V_{ch}$  is the channel potential and  $\gamma$  is the body factor defined by  $\sqrt{2q\epsilon_{Si}N_A/C_{ox}}$ . Here  $N_A$  is acceptor concentration in homogeneously doped channel and  $C_{ox} = \epsilon_{ox}/t_{ox}$  is the oxide capacitance per unit area,  $t_{ox}$  is the gate oxide capacitance per unit area, as  $\epsilon_{ox}$  is the oxide permittivity.

Eq. (9) makes the base of all so-called surface potential based (SPBM) MOSFET models which are the most accurate physically based MOSFET models. However, it is obvious that Eq. (9) can be solved with respect to  $\psi_s$  only numerically, what represent the main drawback of the SPBM from the design and physical point of view. Note that, after some elementary transformations, the Eq. (9) can be rearranged in the following, dimensionless form

$$A \exp\left(\frac{y - C}{u_T}\right) - y^2 + (2x + \gamma^2)y - x^2 = 0, \quad (10)$$

where  $A := \gamma^2 u_T$ ,  $x := V_G - V_{FB}$ ,  $C := 2\phi_F + V_{ch}$  and  $y := \psi_s(x)$  is an unknown function. To find the (approximative) solution of Eq. (10), we apply the HPM technique. Firstly, we construct the homotopy equation

$$(1 - p) \mathcal{L}[Y(x; p)] + p \mathcal{N}[Y(x; p)] = 0, \quad (11)$$

where  $p \in (0, 1)$  is the embedding parameter, as

$$\mathcal{L}[Y(x; p)] = Y(x; p) - f_0(x)$$

is a linear part, and

$$\mathcal{N}[Y(x; p)] = A \exp\left(\frac{Y(x; p) - C}{u_T}\right) - Y^2(x; p) + (2x + \gamma^2)Y(x; p) - x^2$$

is a non-linear (“true”) part of Eq. (10).

When  $p = 0$  the homotopy Eq. (11) obviously becomes  $\mathcal{L}[Y(x; 0)] = 0$ , with the unique initial solution  $Y(x; 0) = f_0(x)$ . Similarly, for  $p = 1$ , Eq. (11) becomes  $\mathcal{N}[Y(x; 1)] = 0$ , and it is equivalent to Eq. (10), with the “main” solution  $Y(x; 1) \equiv \psi_s(x)$ .

Now, according to aforementioned facts, solution of the homotopy Eq. (11) can be expressed as the power series in  $p$ :

$$\widetilde{y}(x; p) = \sum_{k=0}^{\infty} p^k f_k(x). \quad (12)$$

From here, we get  $f_0(x) := Y(x; 0)$  as the initial solution of the Eq. (11), obtained for  $p = 0$ . On the other hand, the “main” solution of the Eq. (11), obtained for  $p = 1$ , will be

$$\psi_s(x) \equiv Y(x; 1) = \lim_{p \rightarrow 1^-} Y(x; p) = \sum_{k=0}^{\infty} f_k(x). \quad (13)$$

Thus, it represents the solution of the Eq. (10), on the condition of convergence the series in (13).

Now, substituting Eq. (12) in the homotopy Eq. (11), and by using Taylor’s expansion of the exponential term, we obtain

$$\begin{aligned} (1 - p) \sum_{k=1}^{\infty} p^k f_k(x) &= p \left\{ A \sum_{j=0}^{\infty} \frac{1}{j!} u_T^j \left[ \sum_{k=0}^{\infty} p^k f_k(x) \right]^j \right. \\ &\quad \left. - \left[ \sum_{k=0}^{\infty} p^k f_k(x) \right]^2 + B(x) \sum_{k=0}^{\infty} p^k f_k(x) - C(x) \right\}, \end{aligned} \quad (14)$$

where we denoted  $B(x) := 2x + \gamma^2 + 1$  and  $C(x) := x^2 + f_0(x)$ . By equating expressions with the identical powers  $p^k$ ,  $k = 1, 2, \dots$

we obtain the explicit expression of the functions  $\{f_k(x)\}$ . They can be expressed, recursively, with the following recurrence relations:

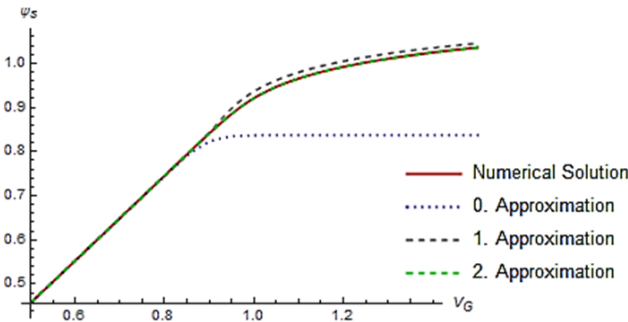
$$\begin{aligned} f_1(x) &= A \exp\left(\frac{f_0(x)}{u_T}\right) - f_0(x)^2 + B(x)f_0(x) - C(x), \\ f_2(x) &= A \exp\left(\frac{f_0(x)}{u_T}\right) \cdot \frac{f_1(x)}{u_T} - 2f_0(x)f_1(x) + B(x)f_1(x), \\ f_3(x) &= A \exp\left(\frac{f_0(x)}{u_T}\right) \cdot \left[\frac{f_2(x)}{u_T} + \frac{f_1(x)^2}{2u_T^2}\right] - [2f_0(x)f_2(x) \\ &\quad + f_1(x)^2] + B(x)f_2(x), \\ &\vdots \end{aligned}$$

In the general case, by using the induction method, it can be easily shown that, for  $k \geq 2$  and  $m! \leq k < (m+1)!$ , hold the following equalities:

$$\begin{aligned} f_k(x) &= A \exp\left(\frac{f_0(x)}{u_T}\right) \cdot \sum_{j=1}^m \frac{1}{j! u_T^j} \sum_{i_1+\dots+i_j=k-1} f_{i_1}(x) \cdots f_{i_j}(x) \\ &\quad - \sum_{j=0}^{k-1} f_j(x)f_{k-j-1}(x) + B(x)f_{k-1}(x). \end{aligned} \quad (15)$$

At last, the equalities above give the appropriate estimates of surface potential  $y = \psi_s(x)$ :

$$\begin{aligned} \widehat{\psi}_s^{(0)}(x) &= f_0(x), \\ \widehat{\psi}_s^{(k)}(x) &= \widehat{\psi}_s^{(k-1)}(x) + f_k(x) = \sum_{j=1}^k f_j(x), \quad k = 1, 2, \dots \end{aligned}$$



**Figure. 1.** The HPM-approximations of the surface potential  $\psi_s(V_G)$ .

Fig.1 shows the surface potential versus gate voltage  $V_G$ . The convergences of the HPM-approximations are illustrated, along with the numerically obtained solution of Eq. (9). For initial solution was used the *the interpolation function*:

$$f_0(x) = \psi_{wi}(x) - u_T \log \left[ 1 + \exp\left(\frac{\psi_{wi}(x) - 2\phi_F - V_{ch}}{x}\right) \right], \quad (16)$$

where

$$\psi_{wi}(x) = \left( -\frac{\gamma}{2} + \sqrt{x + \frac{\gamma^2}{4}} \right)^2 \quad (17)$$

is an approximation of the surface potential  $\psi_s$  in the so-called *weak inversion region* (i.e. when  $\psi_s < 2\phi_F + V_{ch}$ ). As it easily can be seen, already for  $k \geq 2$ , the HPM-approximations  $\{\widehat{\psi}_s^{(k)}(x)\}$  give precise approximations of the surface potential  $\psi_s(x)$ .

#### 4. DETERMINATION OF INVARIANT MEASURES

In the researching of nonlinear chaotic dynamical models special attention is paid to determining their potential stochastic characteristics. Then, the precise analyses of the behavior of these models commonly needs the using of the *invariant (probabilistic) measures (IMs)*. One-dimensional nonlinear chaotic model is an usually defined by the operator  $T : A \rightarrow A$ , where  $A \subseteq \mathbb{R}$ . If chaotic map  $T(x)$  has a finite set of inverse branches  $T^{-1}(x) = \{g_1(x), \dots, g_\ell(x)\}$ , the determination of its IM is based on solving the well-known *Frobenius-Perron equation*:

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}. \quad (18)$$

Here,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is unknown probability density function which corresponds to appropriate IM, i.e. such that, for any Borel set  $B \subseteq \mathbb{R}$ , satisfies the following *T-invariant condition*:

$$\int_B f(x) dx = \int_B f \circ T(x) dx. \quad (19)$$

Since, as it is known, there no exists a general procedure to solve Frobenius-Perron Eq. (18) here is proposed one of the possible way of its solving, based on the HPM. For this purpose, we construct the following *homotopy equation*:

$$(1-p) \left[ F(x; p) - f_0(x) \right] + p \left[ F(x; p) - \sum_{y \in T^{-1}(x)} \frac{F(y; p)}{|T'(y)|} \right] = 0. \quad (20)$$

Here,  $p \in [0, 1]$  is embedding parameter and  $F(x, p)$  is solution of the homotopy Eq. (20) expressed as the power series given by Eq. (6). Thus, the solution of Frobenius-Perron Eq. (18) can be obtain in the same way as in Eq. (7), i.e. by substituting the power series from Eq. (6) in Eq. (20). After some computations, the following equations get ones:

$$\begin{aligned} f_1(x) &= \sum_{y \in T^{-1}(x)} \frac{f_0(y)}{|T'(y)|} - f_0(x), \\ f_k(x) &= \sum_{y \in T^{-1}(x)} \frac{f_{k-1}(y)}{|T'(y)|}, \quad k \geq 2. \end{aligned} \quad (21)$$

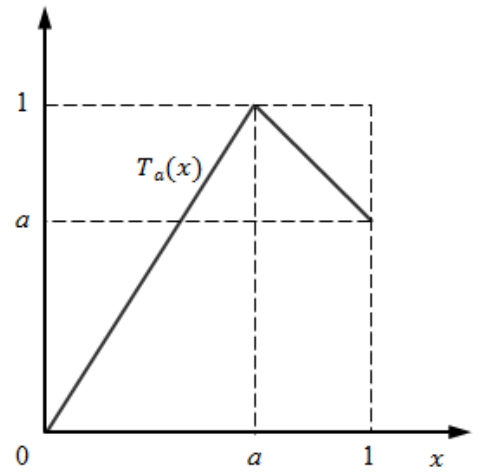
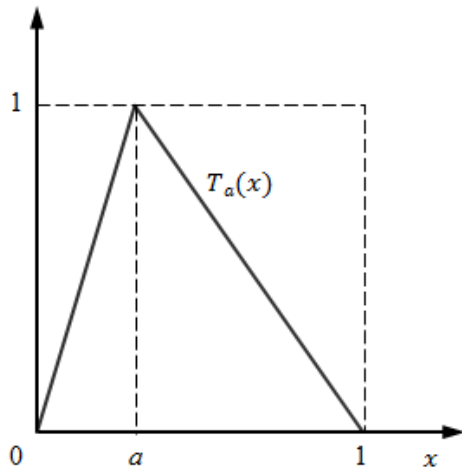
Using Eqs. (21), functions  $\{f_k(x)\}$  can be obtained recursively for an arbitrary  $k = 1, 2, \dots$ . The appropriate *HPM-approximations* of the unknown function  $f(x)$  are

$$\widehat{f}_k(x) := \sum_{j=0}^k f_j(x), \quad k = 0, 1, 2, \dots \quad (22)$$

and their convergence, under some sufficiently conditions, was proven in Stojanović et al. (2018). In the following, some examples of application of the HPM in determining IMs will be described.

**Example 1 ( $\Lambda$ -map).** On the closed unit interval  $[0, 1]$  consider the so-called *Lambda ( $\Lambda$ ) map* (Fig. 2, left panel):

$$T_a(x) = \begin{cases} \frac{x}{a}, & 0 \leq x \leq a, \\ \frac{1-x}{1-a}, & a < x \leq 1, \end{cases}$$



**Figure 2.** Graphs of the  $\Lambda$ -map (panel left) and the truncated  $\Lambda$ -map (panel right).

where  $a \in (0, 1)$  is a predefined parameter. In this case, we have:

$$|T'_a(x)| = \begin{cases} \frac{1}{a}, & 0 \leq x \leq a, \\ \frac{1}{1-a}, & a < x \leq 1, \end{cases}$$

and  $T_a^{-1}(x) = \{ax, 1 - (1-a)x\}$  is the set of inverse branches. Thus, the Frobenius-Perron equation is

$$f(x) = af(ax) + (1-a)f(1 - (1-a)x).$$

If we take, as an initial approximation  $f_0(x) \equiv 1$ , then the first of Eqs. (21) gives:

$$f_1(x) = af_0(ax) + (1-a)f_0(1 - (1-a)x) - f_0(x) \equiv 0.$$

After that, using the second of Eqs. (21), immediately follows  $f_k(x) \equiv 0$ , for any  $k \geq 2$ . In that way, the initial approximation  $f_0(x) \equiv 1$  is exact solution of the Frobenius-Perron Eq. (18), i.e. the invariant probability measure for this map is standardized Lebesgue measure.

**Example 2 (Truncated  $\Lambda$ -map).** Consider again the closed unit interval  $[0, 1]$  and the following map (Fig. 2, right panel):

$$T_a(x) = \begin{cases} \frac{x}{a}, & 0 \leq x < a, \\ -x + a + 1, & a \leq x \leq 1, \end{cases}$$

where, in the same way as in the previous case,  $a \in (0, 1)$  is a predefined parameter. Here, we have:

$$|T'_a(x)| = \begin{cases} \frac{1}{a}, & 0 \leq x < a, \\ 1, & a \leq x \leq 1, \end{cases}$$

and the set of inverse branches is  $T_a^{-1}(x) = \{g_1(x), g_2(x)\}$ , where:

$$g_1(x) = ax, \quad 0 \leq x \leq 1,$$

$$g_2(x) = \begin{cases} 0, & 0 \leq x < a, \\ -x + a + 1, & a \leq x \leq 1, \end{cases}$$

According to this, here the Frobenius-Perron Eq. (18) takes the following form:

$$f(x) = \begin{cases} af(g_1(x)), & 0 \leq x < a, \\ af(g_1(x)) + f(g_2(x)), & a \leq x \leq 1. \end{cases}$$

Now, if we take, as in previous example  $f_0(x) \equiv 1$  and apply Eqs. (21), we obtain:

$$\begin{aligned} f_1(x) &= \begin{cases} af_0(g_1(x)) - f_0(x), & 0 \leq x < a, \\ af_0(g_1(x)) + f_0(g_2(x)) - f_0(x), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a - 1, & 0 \leq x < a, \\ a, & a \leq x \leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} f_2(x) &= \begin{cases} af_1(g_1(x)), & 0 \leq x < a, \\ af_1(g_1(x)) + f_1(g_2(x)), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a(a-1), & 0 \leq x < a, \\ a^2, & a \leq x \leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} f_3(x) &= \begin{cases} af_2(g_1(x)), & 0 \leq x < a, \\ af_2(g_1(x)) + f_2(g_2(x)), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a^2(a-1), & 0 \leq x < a, \\ a^3, & a \leq x \leq 1, \text{ etc.} \end{cases} \end{aligned}$$

In general, using the induction method, it can be easily proven that equalities:

$$f_k(x) = \begin{cases} a^{k-1}(a-1), & 0 \leq x < a, \\ a^k, & a \leq x \leq 1, \end{cases}$$

hold for an arbitrary  $k \geq 1$ . According to thus obtained series  $\{f_k(x)\}$ , the solution of Frobenius-Perron equation in this case is as following:

$$f(x) := \sum_{k=0}^{\infty} f_k(x) = \begin{cases} 1 + (a-1) \sum_{k=1}^{\infty} a^{k-1}, & 0 \leq x < a, \\ \sum_{k=0}^{\infty} a^k, & a \leq x \leq 1, \end{cases}$$

$$= \begin{cases} 0, & 0 \leq x < a, \\ \frac{1}{1-a}, & a \leq x \leq 1, \end{cases}$$

i.e. it concentrated on interval  $(a, 1)$ . It can be easily proved that the last expression in Eq. (23) represents the invariant probability density of truncated  $\Lambda$ -map, i.e. the exact solution of the Frobenius-Perron equation. Also, notice that, according to the well-known facts about geometric series, the above HPM procedure converges for arbitrary  $a \in (0, 1)$ .

## 5. CONCLUSION

This paper describes the Homotopy perturbation method (HPM) representing the powerful technique for solving of nonlinear equations of various kinds. The HPM has been introduced in approximate solving of the well-known implicit relation between the electrostatic surface potential and terminal voltages in MOSFET transistor. As the second application of the HPM has been chosen the determination of the invariant measures (IMs) of the non-linear dynamical systems with chaotic behavior.

In the first case, the simple mathematical form and high degree of accuracy of obtained HPM approximate analytical solution lead to the improved surface potential based MOSFET model preferring for various circuit simulation programs. On the other hand, due to the second application of HPM procedure is confirmed that invariant measures of the chaotic mappings can be easily analytically determined. Two considered examples have been presented to elucidate the efficiency and implementation of the HPM in solving of various kind of research problems.

## References

- Adamu, G., Bawa, M., Jiya, M., & Chado, U. 2017, A mathematical model for the dynamics of Zika virus via homotopy perturbation method, *Journal of Applied Sciences and Environmental Management*, 21 (4), 615–623
- Biazar, J., Badpeima, F., & Azimi, F. 2009, Application of the homotopy perturbation method to Zakharov–Kuznetsov equations, *Computers & Mathematics with Applications*, 58 (11–12), 2391–2394
- Bota, C. & Caruntu, B. 2017, Approximate analytical solutions of nonlinear differential equations using the Least Squares Homotopy Perturbation Method, *Journal of Mathematical Analysis and Applications*, 448, 401–408
- Chen, T. L. & Gildenblat, G. 2001, Analytical approximation for the MOSFET surface potential, *Solid-State Electronics*, 45, 335–339
- Dong, C., Chen, Z., & Jiang, W. 2013, A modified homotopy perturbation method for solving the nonlinear mixed Volterra–Fredholm integral equation, *Journal of Computational and Applied Mathematics*, 239 (1), 359–366

- El-Sayed, A., Elsaid, A., El-Kalla, I., & Hammadb, D. 2012, A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains, *Applied Mathematics and Computation*, 218 (17)
- Gadallah, M. R. & Elzaki, T. M. 2017, An application of improvement of new homotopy perturbation method for solving third order nonlinear singular partial differential equations, *Universal Journal of Mathematics*, 2 (1), 110–124
- He, J.-H. 1999, Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering*, 178 (3–4), 257–262
- He, J.-H. 2000, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *International Journal of Non-Linear Mechanics*, 35 (1), 37–43
- He, J.-H. 2003, Homotopy perturbation method: A new nonlinear analytical technique, *Applied Mathematics and Computation*, 135 (1), 73–79
- He, J.-H. 2006, Homotopy perturbation method for solving boundary value problems, *Physics Letters A*, 350 (1–2), 87–88
- He, J.-H. 2008, Resent development of the homotopy perturbation method, *Topological Methods in Nonlinear Analysis*, 31 (2), 205–209
- Hetmaniok, E., Nowak, I., Slota, D., & Witula, R. 2013, A study of the convergence of error estimation for the homotopy perturbation method for the Volterra–Fredholm integral equations, *Applied Mathematics Letters*, 26 (1), 165–169
- Hetmaniok, E., Slota, D., & Witula, R. 2012, Convergence and error estimation of homotopy perturbation method for Fredholm and Volterra integral equations, *Applied Mathematics and Computation*, 218 (21), 10717–10725
- Kevkić, T., Stojanović, V., & Petković, D. 2018, Solving Schrödinger equation for a particle in one-dimensional lattice: An homotopy perturbations approach, *Romanian Reports in Physics*, in press, accepted manuscript
- Kevkić, T., Stojanović, V., & Randjelović, D. 2017, Application of homotopy perturbation method in solving coupled Schrödinger and Poisson equation in accumulation layer, *Romanian Journal of Physics*, 62 (9–10), Article No. 122
- Khan, M., Farasat Saddiq, S., Khan, S., Islam, S., & Ahmad, F. 2014, Application of homotopy perturbation method to an SIR epidemic model, *Journal of Applied Environmental and Biological Sciences*, 4, 49–54
- Noor, M. & Khan, W. 2012, New iterative methods for solving nonlinear equation by using homotopy perturbation method, *Applied Mathematics and Computation*, 219, 3565–3574
- Stojanović, V., Kevkić, T., Gordana, J., & Randjelović, D. 2018, Determination of invariant measures: An approach based on homotopy perturbations, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, in press, accepted manuscript
- Tripathi, R. & Mishra, H. K. 2016, Homotopy perturbation method with Laplace transform (LT-HPM) for solving Lane–Emden type differential equations (LETDEs) (New York: SpringerPlus)
- van Langevelde, R. & Klaassen, F. 2000, An explicit surface-potential-based MOSFET model for circuit simulation, *Solid-State Electronics*, 44, 409–418
- Zeb, M., Haroon, T., & Siddiqui, A. 2014, Homotopy perturbation solution for flow of a third-grade fluid in helical screw rheometer, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 76 (4), 179–190
- Zhang, M.-F., Liu, Y.-Q., & Zhou, X.-S. 2015, Efficient homotopy perturbation method for fractional non-linear equations using Sumudu transform, *Thermal Science*, 19, 1167–1171